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Momentum transfer at the boundary between a porous medium and a homogeneous fluid—I. Theoretical development

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Abstract—The momentum transfer condition that applies at the boundary between a porous medium and a homogeneous fluid is developed as a jump condition based on the non-local form of the volume averaged momentum equation. Outside the boundary region this non-local form reduces to the classic transport equations, i.e. Darcy's law and Stokes' equations. The structure of the theory is comparable to that used to develop jump conditions at phase interfaces, thus experimental measurements are required to determine the coefficient that appears in the jump condition. The development presented in this work differs from previous studies in that the jump condition is constructed to join Darcy's law *with the Brinkman correction* to Stokes' equations. This approach produces a jump in the stress but not in the velocity, and this has important consequences for heat transfer processes since it allows the convective transport to be continuous at the boundary between a porous medium and a homogeneous fluid.

1. INTRODUCTION

The problem of momentum transport at the boundary between a porous medium and a homogeneous fluid occurs in a wide variety of technological applications and has therefore been the object of a great deal of study. The lubrication process associated with a porous bearing [1] provided the impetus for the original experimental study of Beavers and Joseph [2] in addition to numerous subsequent studies. The fluid mechanical process in the neighborhood of the interface between a spatially periodic porous medium and a homogeneous fluid has been analyzed by Larson and Higdon [3, 4] and by Sahraoui and Kaviany [5] in terms of the point equations, and recent monographs on transport in porous media [6, 7] provide an extensive literature concerning this momentum transport process.

To be specific about the problem under consideration, we refer to Fig. 1, in which we have shown a uniform flow parallel to a fluid-porous medium boundary. The homogeneous fluid occupies the η-region while the porous medium is identified as the ω-region. We will use the phrases, *homogeneous η-region* and *homogeneous ω-region* to refer to those portions of the η- and ω-regions that are not influenced by the rapid changes in structure that occur in the boundary region. The system illustrated in Fig. 1 is analogous to that studied experimentally by Beavers and Joseph [2], and our objective is to develop the appropriate jump condition for momentum transport within the framework of the method of volume averaging. We have shown two averaging volumes in Fig. 1. One of these is spherical and is generally considered appropriate for the study of multi-dimensional transport processes, while the other is planar and represents an acceptable averaging volume for one-dimensional processes. The theory will be presented in a fairly general context; however, the simplifications that can be made for one-dimensional processes are important and one needs to keep in mind the two averaging volumes illustrated in Fig. 1. When the volume averaged equations in the homogeneous η-region are *equivalent to the point equations* we can use standard techniques to develop the jump condition [8].

The governing differential equations and boundary conditions for the momentum transfer process in *both* the ω- and η-regions are given by

$$\nabla \cdot \mathbf{v}_\beta = 0 \quad \text{in the } \beta\text{-phase} \tag{1}$$

$$0 = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta \quad \text{in the } \beta\text{-phase} \tag{2}$$

B.C. 1 $$\mathbf{v}_\beta = 0 \quad \text{at the } \beta\text{–}\sigma \text{ interface} \tag{3}$$

B.C. 2 $$\mathbf{v}_\beta = 0 \quad y = h \tag{4}$$

B.C. 3 $$\mathbf{j} \cdot \langle \mathbf{v}_\beta \rangle = 0 \quad y = -H. \tag{5}$$

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NOMENCLATURE

$A_{\beta\sigma}$ area of the β–σ interface contained within the averaging volume [m^2]
$\mathscr{A}_\infty$ total surface area of the large-scale averaging volume, $\mathscr{V}_\infty$ [m^2]
A_ω that portion of $\mathscr{A}_\infty$ contained in the ω-region [m^2]
A_η that portion of $\mathscr{A}_\infty$ contained in the η-region [m^2]
$A_{\omega\eta}$ area of the ω–η interface contained within $\mathscr{V}_\infty$
$\langle \mathbf{B} \rangle$ excess Brinkman stress [N m^{-2}]
D diameter of the averaging volume having the form of a disk [m]
$\mathbf{g}$ gravity vector [m s^{-2}]
h depth of fluid channel [m]
H depth of porous medium [m]
$\mathbf{I}$ unit tensor
$\mathbf{i}, \mathbf{j}$ unit base vectors in the x- and y-direction, respectively.
$\mathbf{K}_\beta$ Darcy's law permeability tensor [m^2]
$\mathbf{K}_{\beta\omega}$ Darcy's law permeability tensor in the homogeneous ω-region [m^2]
L_v characteristic length associated with the velocity [m]
L_{v1} characteristic length associated with the gradient of the velocity [m]
L_ε characteristic length associated with the porosity [m]
L_{p1} characteristic length associated with the pressure gradient [m]
ℓ_β characteristic length associated with the β-phase in the ω-region [m]
$\mathbf{n}_{\beta\sigma}$ unit normal vector directed from the β-phase toward the σ-phase
$\mathbf{n}_{\omega\eta}$ unit normal vector directed from the ω-region toward the η-region
$\mathbf{P}$ projection tensor
p_β pressure in the β-phase [N m^{-2}]
$\langle p_\beta \rangle^\beta$ intrinsic average pressure [N m^{-2}]
$\langle p_\beta \rangle^\beta_\omega$ intrinsic average pressure in the ω-region [N m^{-2}]
$\langle p_\beta \rangle^\beta_\eta$ intrinsic average pressure in the η-region [N m^{-2}]
$\langle p_\beta \rangle$ $\varepsilon_\beta \langle p_\beta \rangle^\beta$ superficial average pressure [N m^{-2}]
r_0 radius of a spherical averaging volume [m]
$\langle \mathbf{T} \rangle_S$ excess surface stress [N m^{-2}]
$\langle \mathbf{T} \rangle_B$ excess bulk stress [N m^{-2}]
$\mathbf{v}_\beta$ velocity vector in the β-phase [m s^{-1}]
$\mathbf{v}_{\beta\eta}$ fluid velocity vector in the η-region [m s^{-1}]
$\langle \mathbf{v}_\beta \rangle^\beta$ intrinsic average velocity [m s^{-1}]
$\langle \mathbf{v}_\beta \rangle$ $\varepsilon_\beta \langle \mathbf{v}_\beta \rangle^\beta$ superficial average velocity [m s^{-1}]
$\langle \mathbf{v}_\beta \rangle_\omega$ superficial average velocity in the ω-region [m s^{-1}]
$\langle \mathbf{v}_\beta \rangle_\eta$ superficial average velocity in the η-region [m s^{-1}]
$\langle \mathbf{v}_\beta \rangle_s$ excess surface velocity [m s^{-1}]
x, y rectangular coordinates [m]
$\mathbf{x}$ position vector locating the centroid of the averaging volume [m]
$\mathbf{y}$ position vector relative to the centroid of the averaging volume [m]
$\mathscr{V}$ averaging volume [m^3]
$\mathscr{V}_\infty$ large-scale averaging volume [m^3]
V_ω volume of the ω-region contained in $\mathscr{V}_\infty$ [m^3]
V_η volume of the η-region contained in $\mathscr{V}_\infty$ [m^3]
V_β volume of the β-phase contained within the averaging volume $\mathscr{V}$ [m^3].

Greek symbols
β the adjustable coefficient in the representation for the excess stress
Δ thickness of a disk that represents an averaging volume [m]
δ thickness of the interfacial region [m]
ε_β porosity or volume fraction of the β-phase
$\varepsilon_{\beta\omega}$ porosity in the homogeneous portion of the ω-region
λ unit tangent vector to the ω–η interface
μ_β viscosity of the β-phase [Ns m^{-2}]
ρ_β density of the β-phase [kg m^{-3}].

Subscripts
β identifies a quantity associated with the β-phase
$\beta\sigma$ identifies a quantity associated with the β–σ interface
η identifies a quantity associated with the η-region
ω identifies a quantity associated with the ω-region
$\omega\eta$ identifies a quantity associated with the ω–η boundary
ε identifies a length scale associated with the porosity
$p1$ identifies a length scale associated with the gradient of the pressure
s identifies a surface vector or tensor
v identifies a length scale associated with the velocity
$v1$ identifies a length scale associated with the gradient of the velocity
∞ identifies a quantity associated with the large-scale averaging volume.

Superscripts
β identifies an intrinsic volume average.

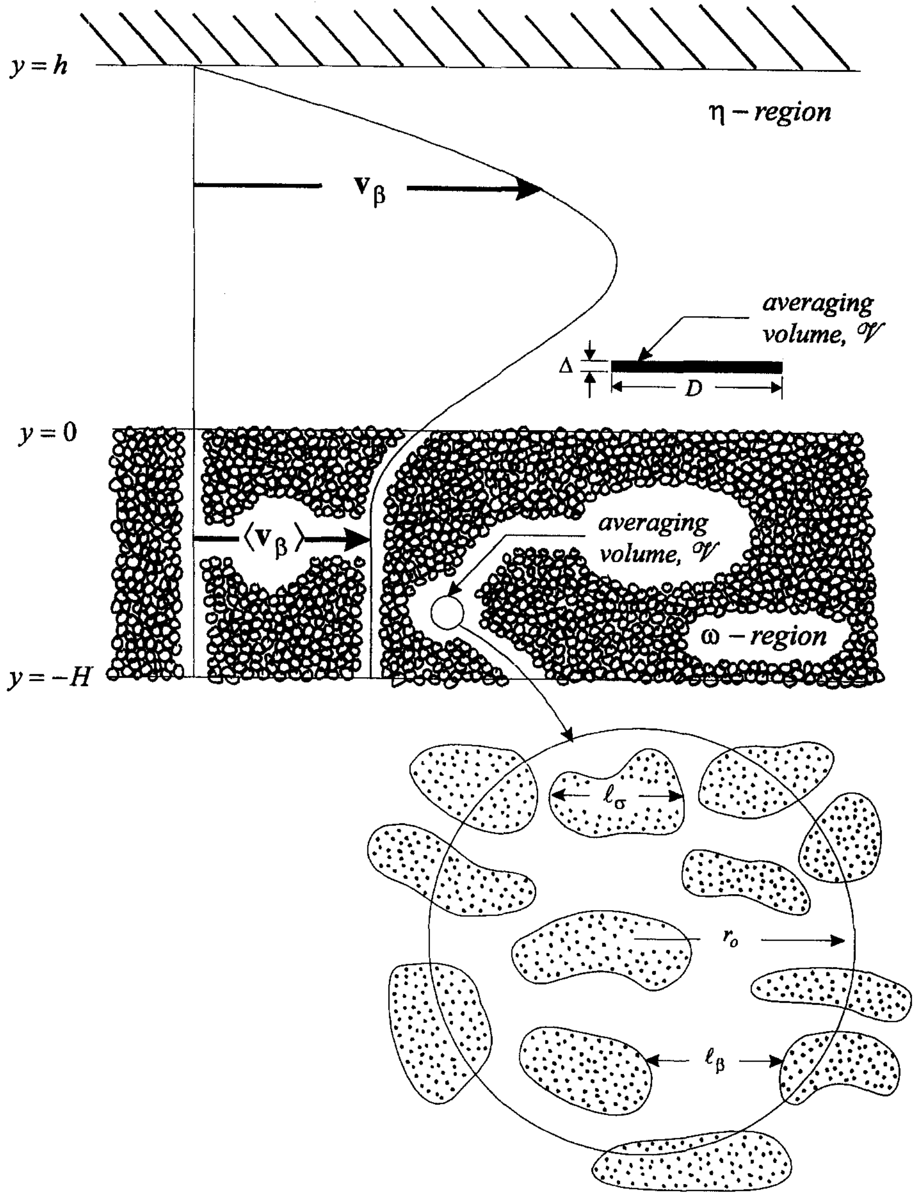

Fig. 1. Flow of a homogeneous fluid parallel to a porous medium.

Here we must remark that, while inertial effects may be negligible in the *homogeneous ω-region* and in the *homogeneous η-region*, it is possible that inertial effects will not be negligible in the boundary region. In that region the curvature of the streamlines will be of the order of the pore or particle diameter and this may lead to non-zero values of the inertial terms, $\rho \mathbf{v} \cdot \nabla \mathbf{v}$.

The boundary condition at $y = -H$ has been expressed by equation (5) in a form that is suitable for use with Darcy's law, thus we have used $\langle \mathbf{v}_\beta \rangle$ to represent the *superficial* volume averaged velocity defined by

$$\langle \mathbf{v}_\beta \rangle|_{\mathbf{x}} = \frac{1}{\mathscr{V}} \int_{V_\beta(\mathbf{x})} \mathbf{v}_\beta(\mathbf{x}+\mathbf{y}_\beta)\, \mathrm{d}V_y. \tag{6}$$

Here $V_\beta(\mathbf{x})$ is the volume of the β-phase contained within the averaged volume illustrated in Fig. 1. The position vectors used in equation (6) are identified in Fig. 2, where we have indicated that $\mathbf{x}$ represents the vector locating the centroid of the averaging volume, and that $\mathbf{y}_\beta$ represents the vector locating points in the β-phase relative to the centroid. Equation (6) clearly indicates that volume averaged quantities are associated with the centroid and that integration is carried out with respect to the components of $\mathbf{y}_\beta$.

The boundary conditions given by equations (4) and (5) are an indication of the *mismatch of length-scales* that one often encounters in transport problems that involve a porous medium. The point boundary

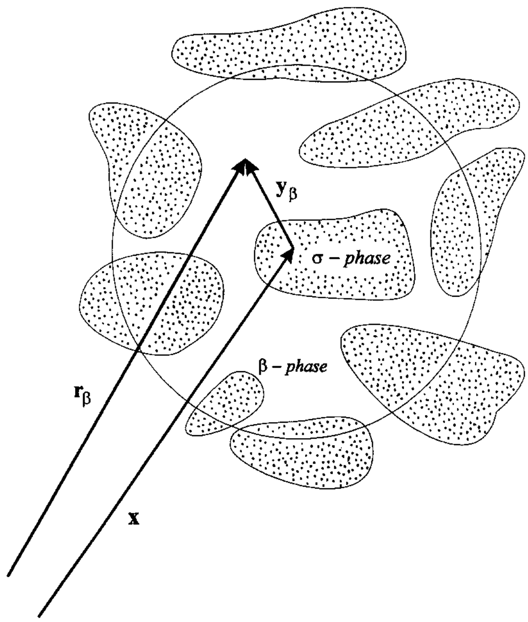

Fig. 2. Position vectors associated with the averaging volume.

condition given by equation (4) is based on the idea that the point velocity is continuous, while the volume averaged boundary condition represented by equation (5) is an approximation based on the idea that the interface at $y = -H$ is impermeable. The mismatch between point and volume averaged quantities at the interface between a porous medium and a homogeneous fluid has been noted by various authors [7, 9, 10] and good arguments have been put forth to support the idea that volume averaged quantities should be continuous at boundaries that involve a porous medium. If one accepts continuity of the volume averaged velocity, it is not illogical to move one step further and make use of volume averaged transport equations in *both* regions adjacent to the boundary between the porous medium and the homogeneous fluid. Under certain circumstances the point and volume averaged transport equations are equivalent and when this occurs the development of a jump condition is greatly simplified.

1.1. *Point and volume averaged quantities*

The volume averaged velocity in the η-region can be expressed as

$$\langle \mathbf{v}_\beta \rangle|_\mathbf{x} = \frac{1}{\mathscr{V}} \int_{\mathscr{V}} \mathbf{v}_\beta(\mathbf{x}+\mathbf{y}_\beta)\, \mathrm{d}V_y \tag{7}$$

provided that the centroid is located far enough away from the boundary region so that the averaging volume contains only the β-phase, i.e. $V_\beta(\mathbf{x}) = \mathscr{V}$. A Taylor series expansion about the centroid allows us to express equation (7) as

$$\langle \mathbf{v}_\beta \rangle|_\mathbf{x} = \frac{1}{\mathscr{V}} \int_{\mathscr{V}} [\mathbf{v}_\beta|_\mathbf{x} + \mathbf{y}_\beta \cdot \nabla \mathbf{v}_\beta|_\mathbf{x} + \tfrac{1}{2}\mathbf{y}_\beta\mathbf{y}_\beta : \nabla\nabla\mathbf{v}_\beta|_\mathbf{x} + \ldots]\, \mathrm{d}V_y \tag{8}$$

and this immediately leads to

$$\langle \mathbf{v}_\beta \rangle|_\mathbf{x} = \mathbf{v}_\beta|_\mathbf{x} + \langle \mathbf{y}_\beta \rangle \cdot \nabla \mathbf{v}_\beta|_\mathbf{x} + \tfrac{1}{2}\langle \mathbf{y}_\beta\mathbf{y}_\beta \rangle : \nabla\nabla\mathbf{v}_\beta|_\mathbf{x} + \tfrac{1}{6}\langle \mathbf{y}_\beta\mathbf{y}_\beta\mathbf{y}_\beta \rangle \vdots \nabla\nabla\nabla\mathbf{v}_\beta|_\mathbf{x} + \ldots \tag{9}$$

Since $\mathbf{y}_\beta$ is the position vector relative to the centroid of $\mathscr{V}$ we have *by definition*

$$\langle \mathbf{y}_\beta \rangle = 0 \tag{10}$$

and if the averaging volume is a sphere or a disk as suggested in Fig. 1 we know that $\langle \mathbf{y}_\beta\mathbf{y}_\beta\mathbf{y}_\beta \rangle$ will also be zero. Thus, for most transport processes in homogeneous fluids, equation (9) reduces to

$$\langle \mathbf{v}_\beta \rangle|_\mathbf{x} = \mathbf{v}_\beta|_\mathbf{x} + \tfrac{1}{2}\langle \mathbf{y}_\beta\mathbf{y}_\beta \rangle : \nabla\nabla\mathbf{v}_\beta|_\mathbf{x}. \tag{11}$$

It should be clear that $\langle \mathbf{y}_\beta\mathbf{y}_\beta \rangle$ is on the order of r_0^2 for a spherical averaging volume, thus we can express equation (11) as

$$\langle \mathbf{v}_\beta \rangle|_\mathbf{x} = \mathbf{v}_\beta|_\mathbf{x} + \mathbf{O}\left(\frac{r_0^2}{L_{v1} L_v}\right)\mathbf{v}_\beta|_\mathbf{x} \tag{12}$$

in which L_v and L_{v1} are characteristic lengths defined by

$$\nabla \mathbf{v}_\beta = \mathbf{O}\left(\frac{\Delta \mathbf{v}_\beta}{L_v}\right) \quad \nabla(\nabla \mathbf{v}_\beta) = \mathbf{O}\left(\frac{\Delta(\nabla \mathbf{v}_\beta)}{L_{v1}}\right). \tag{13}$$

Here we have used $\Delta \mathbf{v}_\beta$ to represent the change in velocity that occurs over the distance L_v, and we have used $\Delta(\nabla \mathbf{v}_\beta)$ to represent the change in the velocity gradient that occurs over the distance L_{v1}. From equation (12) we see that the point velocity is equal to the volume average velocity in the η-region

$$\langle \mathbf{v}_\beta \rangle|_\mathbf{x} = \mathbf{v}_\beta|_\mathbf{x} \quad \text{in the homogeneous } \eta\text{-region} \tag{14}$$

when the following length-scale constraint is satisfied

$$\frac{r_0^2}{L_{v1} L_v} \ll 1 \quad \text{in the homogeneous } \eta\text{-region}. \tag{15}$$

One should keep in mind that this type of analysis does *not apply* to the ω-region; however, it *does indeed* apply to the η-region and when equation (15) is satisfied the analysis of the jump condition is greatly simplified because a single volume averaged transport equation is valid in both the η- and ω-regions.

For the process illustrated in Fig. 1, the length-scale constraint given by equation (15) takes the form

$$\frac{r_0^2}{h^2} \ll 1 \tag{16}$$

and if we *do not* impose this constraint we cannot approximate the *volume averaged velocity* by the *point velocity* in the η-region. For the particular flow illustrated in Fig. 1, the resolution of this dilemma is the

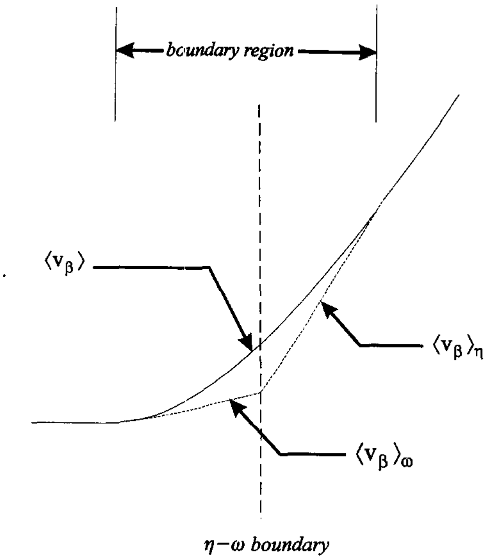

Fig. 3. Velocity profiles in the neighborhood of the interfacial region.

use of a thin disk oriented parallel to the ω–η boundary. Under these circumstances the last term in equation (11) can be estimated as

$$\tfrac{1}{2}\langle \mathbf{y}_\beta \mathbf{y}_\beta \rangle : \nabla\nabla \mathbf{v}_\beta = O\left[\Delta^2 \left(\frac{\partial^2 \mathbf{v}_\beta}{\partial y^2}\right)\right] \tag{17}$$

and we see that the volume averaged velocity will be equal to the point velocity in the homogeneous η-region whenever the following constraint is satisfied.

$$\Delta^2 \ll h^2. \tag{18}$$

While this constraint is easily satisfied in a theoretical sense, it may be difficult if not impossible to accomplish with an experimental technique such as nuclear magnetic resonance imaging [11, 12]. To be precise about the problem under consideration, we have sketched velocity profiles for $\langle v_\beta \rangle = \mathbf{i} \cdot \langle \mathbf{v}_\beta \rangle$ in Fig. 3 for the case in which the point velocity is equal to the average velocity in the homogeneous η-region. Here it becomes clear that we are developing a *jump condition in the stress and requiring the velocity to be continuous* at the ω–η interface. In the next section we will develop the generally valid volume averaged momentum equation which describes $\langle \mathbf{v}_\beta \rangle$, in addition to the special forms that are used to calculated $\langle \mathbf{v}_\beta \rangle_\omega$ and $\langle \mathbf{v}_\beta \rangle_\eta$.

2. VOLUME AVERAGING

In addition to the superficial volume averaged velocity defined by equation (7) we will need to make use of the intrinsic average velocity which is defined according to

$$\langle \mathbf{v}_\beta \rangle^\beta|_\mathbf{x} = \frac{1}{V_\beta} \int_{V_\beta(\mathbf{x})} \mathbf{v}_\beta(\mathbf{x}+\mathbf{y}_\beta)\, \mathrm{d}V_y. \tag{19}$$

The superficial and intrinsic velocities are related by

$$\langle \mathbf{v}_\beta \rangle = \varepsilon_\beta \langle \mathbf{v}_\beta \rangle^\beta \tag{20}$$

in which it is understood that all average quantities are evaluated at the centroid located by $\mathbf{x}$. In equation (20) we have used ε_β to represent the porosity defined explicitly by

$$\varepsilon_\beta = V_\beta / \mathscr{V}. \tag{21}$$

2.1. *Continuity equation*

We begin our analysis with the continuity equation given by equation (1) and form the superficial average to obtain

$$\langle \nabla \cdot \mathbf{v}_\beta \rangle = 0. \tag{22}$$

In order to interchange differentiation and integration in equation (22) we need to make use of the spatial averaging theorem [13] which can be expressed as

$$\langle \nabla \Psi_\beta \rangle = \nabla \langle \Psi_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \Psi_\beta \, \mathrm{d}A \tag{23}$$

and when the vector form of this theorem is used with equation (22) we obtain

$$\nabla \cdot \langle \mathbf{v}_\beta \rangle + \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \mathbf{v}_\beta \, \mathrm{d}A = 0. \tag{24}$$

Imposition of the no-slip condition given by equation (3) allows us to express this result as

$$\nabla \cdot \langle \mathbf{v}_\beta \rangle = 0 \tag{25}$$

and we see that the superficial volume averaged velocity field is solenoidal. The superficial velocity is preferred for solving problems because of its solenoidal characteristic; however there is often considerable confusion in the literature concerning the superficial and intrinsic velocities and in this work we will always be careful to distinguish these two velocities in terms of the nomenclature illustrated by equation (20).

2.2. *Momentum equation*

The superficial volume average of the Stokes' equations can be expressed as

$$0 = -\langle \nabla p_\beta \rangle + \langle \rho_\beta \mathbf{g} \rangle + \langle \mu_\beta \nabla^2 \mathbf{v}_\beta \rangle. \tag{26}$$

It is permissible to ignore variations in ρ_β and μ_β *within the averaging volume* so that this result takes the form

$$0 = -\langle \nabla p_\beta \rangle + \varepsilon_\beta \rho_\beta \mathbf{g} + \mu_\beta \langle \nabla^2 \mathbf{v}_\beta \rangle \tag{27}$$

and we can use the averaging theorem twice in order to obtain [14]

$$0 = -\nabla \langle p_\beta \rangle + \varepsilon_\beta \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot [-I p_\beta + \mu_\beta \nabla \mathbf{v}_\beta] \, \mathrm{d}A. \tag{28}$$

Here one might want to note that the third term in this result represents the Brinkman [15] correction and that the viscosity associated with this term is the fluid viscosity, μ_β, which need not be *corrected* in any way.

While the *superficial velocity* is preferred because of its solenoidal characteristic, the *intrinsic pressure* is preferred because it more closely resembles the pressure that one could measure or the pressure that one could impose as a boundary condition. This requires that we make use of the analogous form of equation (20) for the pressure

$$\langle p_\beta \rangle = \varepsilon_\beta \langle p_\beta \rangle^\beta \tag{29}$$

in order to express equation (28) as

$$0 = -\varepsilon_\beta \nabla \langle p_\beta \rangle^\beta - \langle p_\beta \rangle^\beta \nabla \varepsilon_\beta + \varepsilon_\beta \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot [-Ip_\beta + \mu_\beta \nabla \mathbf{v}_\beta] \, \mathrm{d}A. \tag{30}$$

This result can be simplified by means of a lemma obtained from equation (23) by letting Ψ_β be a constant. This lemma takes the form

$$\nabla \varepsilon_\beta = -\frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \, \mathrm{d}A \tag{31}$$

and it allows us to express $\langle p_\beta \rangle^\beta \nabla \varepsilon_\beta$ as

$$\langle p_\beta \rangle^\beta \nabla \varepsilon_\beta = -\frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \langle p_\beta \rangle^\beta |_{\mathbf{x}} \, \mathrm{d}A. \tag{32}$$

When volume averaged quantities appear in governing differential equations such as equation (25) or (28), it is understood that they are evaluated at the centroid of the averaging volume; however, when they appear inside integrals confusion can result, thus we have been careful to note that $\langle p_\beta \rangle^\beta$ is evaluated at the centroid on the right hand side of equation (32). We can also use equation (31) with the gradient of the intrinsic velocity to obtain

$$\nabla \varepsilon_\beta \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta = -\frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta |_{\mathbf{x}} \, \mathrm{d}A \tag{33}$$

and when these two results are used with equation (30) the volume averaged Stokes' equations take the form [16, 17]

$$\begin{aligned} 0 = & -\varepsilon_\beta \nabla \langle p_\beta \rangle^\beta + \varepsilon_\beta \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle \\ & - \mu_\beta (\nabla \varepsilon_\beta) \cdot [\nabla (\varepsilon_\beta^{-1} \langle \mathbf{v}_\beta \rangle)] \\ & + \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot [-\mathbf{I}(p_\beta|_{\mathbf{x}+\mathbf{y}_\beta} - \langle p_\beta \rangle^\beta |_{\mathbf{x}}) \\ & + \mu_\beta (\nabla \mathbf{v}_\beta |_{\mathbf{x}+\mathbf{y}_\beta} - \nabla \langle \mathbf{v}_\beta \rangle^\beta |_{\mathbf{x}})] \, \mathrm{d}A. \end{aligned} \tag{34}$$

Here we have been forced to introduce the intrinsic average velocity in order to arrange the integral in equation (30) in terms of quantities that appear in the *closure problem* for Darcy's law [14]. The first viscous term that appears in equation (34) is usually referred to as the *Brinkman correction*, and this term is often included in the analysis of flow in the boundary region between a porous medium and a homogeneous fluid or in the boundary region between a porous medium and a homogeneous solid. In such regions the second viscous term is the same order of magnitude as the Brinkman correction and one is not justified in neglecting $\mu_\beta(\nabla \varepsilon_\beta) \cdot [\nabla(\varepsilon_\beta^{-1} \langle \mathbf{v}_\beta \rangle)]$. We will refer to this latter term as the *second Brinkman correction.* A key point to remember about equation (34) is that *no length-scale constraints have been imposed* and this means that it is valid *everywhere* in the system illustrated in Fig. 1.

At this point it is convenient to divide equation (34) by ε_β and express the result in a compact form

$$\begin{aligned} 0 = & -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \varepsilon_\beta^{-1} \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle \\ & - \mu_\beta \varepsilon_\beta^{-1} (\nabla \varepsilon_\beta) \cdot [\nabla (\varepsilon_\beta^{-1} \langle \mathbf{v}_\beta \rangle)] - \mu_\beta \mathbf{\Phi}_\beta \end{aligned} \tag{35}$$

in which the vector $\mathbf{\Phi}_\beta$ is defined by

$$\begin{aligned} \mu_\beta \mathbf{\Phi}_\beta = & -\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot [-\mathbf{I}(p_\beta|_{\mathbf{x}+\mathbf{y}_\beta} - \langle p_\beta \rangle^\beta |_{\mathbf{x}}) \\ & + \mu_\beta (\nabla \mathbf{v}_\beta |_{\mathbf{x}+\mathbf{y}_\beta} - \nabla \langle \mathbf{v}_\beta \rangle^\beta |_{\mathbf{x}})] \, \mathrm{d}A. \end{aligned} \tag{36}$$

It has been shown elsewhere [14] that $\mathbf{\Phi}_\beta$ has an especially simple form in a homogeneous porous medium provided that certain length-scale constraints are satisfied. In order to identify the length-scale constraints that must be imposed in order for $\mathbf{\Phi}_\beta$ to have a simple form, we define three length scales in terms of the following estimates

$$\begin{aligned} \nabla \varepsilon_\beta &= \mathbf{O}\left(\frac{\Delta \varepsilon_\beta}{L_\varepsilon}\right) \\ \nabla \nabla \langle p_\beta \rangle^\beta &= \mathbf{O}\left(\frac{\Delta(\nabla \langle p_\beta \rangle^\beta)}{L_{p1}}\right) \\ \nabla \nabla \nabla \langle \mathbf{v}_\beta \rangle^\beta &= \mathbf{O}\left(\frac{\Delta(\nabla \nabla \langle \mathbf{v}_\beta \rangle^\beta)}{L_{v2}}\right). \end{aligned} \tag{37}$$

These estimates are consistent with those given earlier by equation (13) and when the following three length-scale constraints are imposed

$$\frac{r_0^2}{L_\varepsilon L_{p1}} \ll 1 \quad \frac{r_0^2}{L_\varepsilon L_{v2}} \ll 1 \quad \ell_\beta \ll r_0 \tag{38}$$

one can prove [14, 16, 17] that $\mathbf{\Phi}_\beta$ is given by

$$\mathbf{\Phi}_\beta = \mathbf{K}_{\beta\omega}^{-1} \cdot \langle \mathbf{v}_\beta \rangle \quad \text{in the homogeneous } \omega\text{-region.} \tag{39}$$

The first two constraints given in equation (38) are analytical in nature [16–18] and from those developments one can deduce that r_0 should be replaced by Δ when the disk illustrated in Fig. 1 is used in the averaging process. The third constraint in equation (38) is more intuitive in nature [19]; however, recent numerical simulations [20–22] have verified that constraint for spherical averaging volumes. On the basis of both analysis and intuition, we are inclined to

$$\mathscr{A}_\infty = A_\omega + A_\eta. \tag{51}$$

The volumes of the ω- and η-regions contained in $\mathscr{V}_\infty$ are designated by V_ω and V_η and when equations (45) and (46) are integrated over these regions we can express the results as

$$\int_{A_\omega} \mathbf{n}_\omega \cdot \langle \mathbf{v}_\beta \rangle_\omega \, dA + \int_{A_{\omega\eta}} \mathbf{n}_{\omega\eta} \cdot \langle \mathbf{v}_\beta \rangle_\omega \, dA = 0 \tag{52}$$

$$\int_{A_\eta} \mathbf{n}_\eta \cdot \langle \mathbf{v}_\beta \rangle_\eta \, dA + \int_{A_{\eta\omega}} \mathbf{n}_{\eta\omega} \cdot \langle \mathbf{v}_\beta \rangle_\eta \, dA = 0. \tag{53}$$

Here we have used $A_{\omega\eta} = A_{\eta\omega}$ to represent the area of the *dividing surface* contained within the volume $\mathscr{V}_\infty$, and the convention associated with the unit normal vectors at the dividing surface is that $\mathbf{n}_{\omega\eta} = -\mathbf{n}_{\eta\omega}$.

Returning to equations (52) and (53), it is important to remember that the values of $\langle \mathbf{v}_\beta \rangle_\omega$ and $\langle \mathbf{v}_\beta \rangle_\eta$ in the interfacial region are not necessarily equal to the actual physical value which is represented everywhere by $\langle \mathbf{v}_\beta \rangle$. If we subtract equations (52) and (53) from equation (50), we can arrange the result in the following form

$$\int_{A_{\omega\eta}} \mathbf{n}_{\omega\eta} \cdot (\langle \mathbf{v}_\beta \rangle_\omega - \langle \mathbf{v}_\beta \rangle_\eta) \, dA = \int_{A_\omega} \mathbf{n}_\omega \cdot (\langle \mathbf{v}_\beta \rangle - \langle \mathbf{v}_\beta \rangle_\omega) \, dA + \int_{A_\eta} \mathbf{n}_\eta \cdot (\langle \mathbf{v}_\beta \rangle - \langle \mathbf{v}_\beta \rangle_\eta) \, dA \tag{54}$$

and this suggests that we define an *excess surface velocity* according to

3.2. *Definition*

$$\oint_C \mathbf{n}_s \cdot (\delta \langle \mathbf{v}_\beta \rangle_s) \, d\sigma = \int_{A_\omega} \mathbf{n}_\omega \cdot (\langle \mathbf{v}_\beta \rangle - \langle \mathbf{v}_\beta \rangle_\omega) \, dA + \int_{A_\eta} \mathbf{n}_\eta \cdot (\langle \mathbf{v}_\beta \rangle - \langle \mathbf{v}_\beta \rangle_\eta) \, dA. \tag{55}$$

Here C represents a closed curve lying on the dividing surface and σ represents the arc length along this curve. If the thickness δ is specified, one can think of equation (55) as defining the excess surface velocity $\langle \mathbf{v}_\beta \rangle_s$; otherwise, one must think of this relation as a definition of the product, $\delta \langle \mathbf{v}_\beta \rangle_s$. Use of this relation with equation (54) leads to

$$\int_{A_{\omega\eta}} \mathbf{n}_{\omega\eta} \cdot (\langle \mathbf{v}_\beta \rangle_\omega - \langle \mathbf{v}_\beta \rangle_\eta) \, dA - \oint_C \mathbf{n}_s \cdot (\delta \langle \mathbf{v}_\beta \rangle_s) \, d\sigma = 0 \tag{56}$$

and we can apply the surface divergence theorem [23] to obtain

$$\int_{A_{\omega\eta}} [\mathbf{n}_{\omega\eta} \cdot (\langle \mathbf{v}_\beta \rangle_\omega - \langle \mathbf{v}_\beta \rangle_\eta) - \nabla_s \cdot (\delta \langle \mathbf{v}_\beta \rangle_s)] \, dA = 0. \tag{57}$$

In equation (56) we have used $\mathbf{n}_s$ to represent the unit normal vector that is *tangential* to the dividing surface and *normal* to the curve C, while ∇_s has been used to represent the surface gradient operator that is given explicitly in terms of the gradient operator and the projection tensor according to

$$\nabla_s = \mathbf{P} \cdot \nabla, \quad \mathbf{P} = \mathbf{I} - \mathbf{n}_{\omega\eta}\mathbf{n}_{\omega\eta}. \tag{58}$$

Since the limits of integration over $A_{\omega\eta}$ are arbitrary, we extract the following jump condition from equation (57)

$$(\langle \mathbf{v}_\beta \rangle_\omega - \langle \mathbf{v}_\beta \rangle_\eta) \cdot \mathbf{n}_{\omega\eta} = \nabla_s \cdot (\delta \langle \mathbf{v}_\beta \rangle_s) \quad \text{at the } \omega\text{–}\eta \text{ interface}. \tag{59}$$

While the surface velocity, $\langle \mathbf{v}_\beta \rangle_s$, may be important at the boundary between a porous medium and a homogeneous solid [9], and is certainly important at a fracture between two porous media [24], it would appear to be unimportant at the fluid–porous medium interface illustrated in Fig. 1.

3.3. *Momentum equation*

In order to develop the momentum jump condition, we recall that the following volume averaged momentum equation

$$0 = -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \varepsilon_\beta^{-1} \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle - \mu_\beta \varepsilon_\beta^{-1} (\nabla \varepsilon_\beta) \cdot [\nabla (\varepsilon_\beta^{-1} \langle \mathbf{v}_\beta \rangle)] - \mu_\beta \mathbf{\Phi}_\beta \tag{60}$$

is valid *everywhere*, and that in the ω- and η-regions the momentum equations are given by

$$\omega\text{-region:} \quad 0 = -\nabla \langle p_\beta \rangle_\omega^\beta + \rho_\beta \mathbf{g} + \varepsilon_{\beta\omega}^{-1} \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle_\omega - \mu_\beta \mathbf{K}_{\beta\omega}^{-1} \cdot \langle \mathbf{v}_\beta \rangle_\omega \tag{61}$$

$$\eta\text{-region:} \quad 0 = -\nabla \langle p_\beta \rangle_\eta^\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle_\eta. \tag{62}$$

To derive the momentum jump condition, we follow the procedure given by equations (49)–(59); however, in order to integrate equation (60) over the volume $\mathscr{V}_\infty$ we need to arrange the Brinkman correction term into the form of a divergence and this leads to

$$0 = -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla \cdot (\varepsilon_\beta^{-1} \nabla \langle \mathbf{v}_\beta \rangle) + \mu_\beta \varepsilon_\beta^{-1} (\nabla \ln \varepsilon_\beta)^2 \langle \mathbf{v}_\beta \rangle - \mu_\beta \mathbf{\Phi}_\beta. \tag{63}$$

Here we have adopted the nomenclature indicated by

$$(\nabla \ln \varepsilon_\beta)^2 = (\nabla \ln \varepsilon_\beta) \cdot (\nabla \ln \varepsilon_\beta). \tag{64}$$

In equation (61) we need not rearrange the first Brinkman correction since $\varepsilon_{\beta\omega}$ is a constant in that equation.

To develop the jump condition we form the integral of equation (63) over $\mathscr{V}_\infty$ and the integrals of equations (61) and (62) over V_ω and V_η, respectively. We then subtract the latter from the former to obtain impose the following constraints for one-dimensional processes (in the sense of averages).

$$\frac{\Delta^2}{L_\varepsilon L_{p1}} \ll 1 \quad \frac{\Delta^2}{L_\varepsilon L_{v2}} \ll 1 \quad \ell_\beta \ll D. \tag{40}$$

Here Δ represents the thickness of the disk illustrated in Fig. 1 while D represents the diameter. When the constraints indicated by either equation (38) or (40) are valid, the viscous term involving the gradient of the porosity (the second Brinkman correction) can obviously be set equal to zero since it will be negligible compared to the first Brinkman correction.

$$\mu_\beta \varepsilon_\beta^{-1}(\nabla \varepsilon_\beta)\cdot[\nabla(\varepsilon_\beta^{-1}\langle \mathbf{v}_\beta\rangle)] = 0 \quad \text{in the homogeneous } \omega\text{-region.} \tag{41}$$

In equation (39) we have used $\mathbf{K}_{\beta\omega}$ to represent the Darcy's law permeability tensor and the use of equations (39) and (41) in equation (31) leads to

$$\langle \mathbf{v}_\beta\rangle_\omega = -\frac{\mathbf{K}_{\beta\omega}}{\mu_\beta}\cdot(\nabla\langle p_\beta\rangle_\omega^\beta - \rho_\beta \mathbf{g} - \varepsilon_\beta^{-1}\mu_\beta \nabla^2\langle \mathbf{v}_\beta\rangle_\omega) \quad \text{in the homogeneous } \omega\text{-region.} \tag{42}$$

This is Darcy's law with the first Brinkman correction; however, when the length-scale constraints indicated by equation (38) are in effect the first Brinkman correction makes a negligible contribution to equation (42).

3. JUMP CONDITION

In this analysis we consider the large-scale averaging volume illustrated in Fig. 4 and note that the following forms of the volume averaged continuity and momentum equations are valid *everywhere* in the region under consideration.

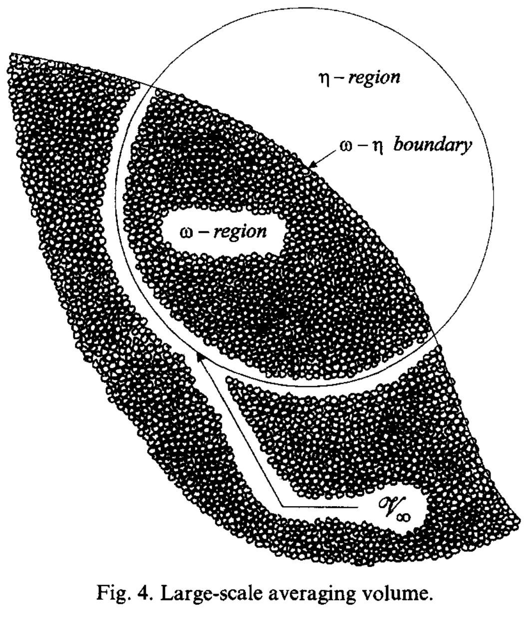

Fig. 4. Large-scale averaging volume.

$$\nabla\cdot\langle \mathbf{v}_\beta\rangle = 0 \tag{43}$$

$$0 = -\nabla\langle p_\beta\rangle^\beta + \rho_\beta \mathbf{g} + \varepsilon_\beta^{-1}\mu_\beta\nabla^2\langle \mathbf{v}_\beta\rangle - \mu_\beta\varepsilon_\beta^{-1}(\nabla\varepsilon_\beta)\cdot[\nabla(\varepsilon_\beta^{-1}\langle \mathbf{v}_\beta\rangle)] - \mu_\beta \mathbf{\Phi}_\beta. \tag{44}$$

One should remember that *no length-scale constraints* have been imposed on these two equations, thus they are limited only by the form of the original point equations and boundary condition given by equation (3). The length-scale constraint indicated by equation (15) is required only to ensure that $\mathbf{v}_{\beta\eta} = \langle \mathbf{v}_\beta\rangle_\eta$ in the homogeneous η-region so that the boundary condition indicated by equation (4) can be employed directly. For one-dimensional processes the single length-scale constraint imposed on this analysis is given by equation (18).

Solution of these two equations to produce the velocity and pressure fields would require a closed form of equation (44) and this can be avoided if an acceptable jump condition can be constructed. In the *homogeneous parts* of the ω- and η-regions we represent the governing equations as

$$\nabla\cdot\langle \mathbf{v}_\beta\rangle_\omega = 0 \quad \text{in the } \omega\text{-region} \tag{45}$$

$$0 = -\nabla\langle p_\beta\rangle_\omega^\beta + \rho_\beta\mathbf{g} + \varepsilon_{\beta\omega}^{-1}\mu_\beta\nabla^2\langle \mathbf{v}_\beta\rangle_\omega - \mu_\beta\mathbf{K}_{\beta\omega}^{-1}\cdot\langle \mathbf{v}_\beta\rangle_\omega \quad \text{in the } \omega\text{-region} \tag{46}$$

$$\nabla\cdot\langle \mathbf{v}_\beta\rangle_\eta = 0 \quad \text{in the } \eta\text{-region} \tag{47}$$

$$0 = -\nabla\langle p_\beta\rangle_\eta^\beta + \rho_\beta\mathbf{g} + \mu_\beta\nabla^2\langle \mathbf{v}_\beta\rangle_\eta \quad \text{in the } \eta\text{-region.} \tag{48}$$

In addition to using these equations outside the boundary region illustrated in Fig. 3, we will also use them *inside* the boundary region. When they are used inside the boundary region, quantities such as $\langle \mathbf{v}_\beta\rangle_\omega$ and $\langle p_\beta\rangle_\eta^\beta$ may not accurately predict the local volume averaged velocity and pressure. The errors generated by the use of equations (45)–(48) inside the boundary region will be corrected by means of the jump condition which ensures that equations (43) and (44) are satisfied *on average*.

3.1. *Continuity equation*

We begin the analysis of the jump conditions with the continuity equation and form the integral of equation (43) over the volume $\mathscr{V}_\infty$ that is illustrated in Fig. 4. This leads to the integral condition

$$\int_{\mathscr{V}_\infty} \nabla\cdot\langle \mathbf{v}_\beta\rangle\,\mathrm{d}V = 0 \tag{49}$$

which requires that equation (43) *be satisfied on average*. We can use the divergence theorem to express this result as

$$\int_{\mathscr{A}_\infty} \mathbf{n}\cdot\langle \mathbf{v}_\beta\rangle\,\mathrm{d}A = 0. \tag{50}$$

The area $\mathscr{A}_\infty$ can also be represented in terms of the bounding surfaces in the ω- and η-regions according to

$$\int_{A_{\omega\eta}} \mathbf{n}_{\omega\eta} \cdot [-\mathbf{I}(\langle p_\beta\rangle_\omega^\beta - \langle p_\beta\rangle_\eta^\beta) + \mu_\beta(\varepsilon_{\beta\omega}^{-1}\nabla\langle \mathbf{v}_\beta\rangle_\omega - \nabla\langle \mathbf{v}_\beta\rangle_\eta)]\,\mathrm{d}A$$
$$= \int_{A_\omega} \mathbf{n}_\omega \cdot [-\mathbf{I}(\langle p_\beta\rangle^\beta - \langle p_\beta\rangle_\omega^\beta) + \mu_\beta(\varepsilon_\beta^{-1}\nabla\langle \mathbf{v}_\beta\rangle - \varepsilon_{\beta\omega}^{-1}\nabla\langle \mathbf{v}_\beta\rangle_\omega)]\,\mathrm{d}A$$
$$+ \int_{A_\eta} \mathbf{n}_\eta \cdot [-\mathbf{I}(\langle p_\beta\rangle^\beta - \langle p_\beta\rangle_\eta^\beta) + \mu_\beta(\varepsilon_\beta^{-1}\nabla\langle \mathbf{v}_\beta\rangle - \nabla\langle \mathbf{v}_\beta\rangle_\eta)]\,\mathrm{d}A$$
$$- \int_{V_\omega} \mu_\beta[\mathbf{\Phi}_\beta - \mathbf{K}_{\beta\omega}^{-1} \cdot \langle \mathbf{v}_\beta\rangle_\omega]\,\mathrm{d}V - \int_{V_\eta} \mu_\beta[\mathbf{\Phi}_\beta]\,\mathrm{d}V$$
$$+ \int_{V_\omega} [\mu_\beta\varepsilon_\beta^{-1}(\nabla \ln \varepsilon_\beta)^2\langle \mathbf{v}_\beta\rangle]\,\mathrm{d}V + \int_{V_\eta} [\mu_\beta\varepsilon_\beta^{-1}(\nabla \ln \varepsilon_\beta)^2\langle \mathbf{v}_\beta\rangle]\,\mathrm{d}V. \quad (65)$$

Each one of the integrals on the right hand side of this result contains an integrand that tends to zero in the homogeneous regions. For example, we know that $\mathbf{\Phi}_\beta$ has the characteristics given by

$$\mathbf{\Phi}_\beta = \begin{cases} \mathbf{K}_{\beta\omega}^{-1} \cdot \langle \mathbf{v}_\beta\rangle_\omega & \text{in the homogeneous } \omega\text{-region} \\ 0 & \text{in the homogeneous } \eta\text{-region} \end{cases} \quad (66)$$

so that the integrands in the integrals over V_ω and V_η both tend to zero in the homogeneous parts of the ω- and η-regions, respectively. This means that the terms on the right hand side of equation (65) can be represented in terms of *excess functions*.

3.4. *Definitions*

We define an *excess surface stress* according to

$$\oint_C \mathbf{n}_s \cdot \delta\langle \mathbf{T}\rangle_S\,\mathrm{d}\sigma = \int_{A_\omega} \mathbf{n}_\omega \cdot [-\mathbf{I}(\langle p_\beta\rangle^\beta - \langle p_\beta\rangle_\omega^\beta) + \mu_\beta(\varepsilon_\beta^{-1}\nabla\langle \mathbf{v}_\beta\rangle - \varepsilon_{\beta\omega}^{-1}\nabla\langle \mathbf{v}_\beta\rangle_\omega)]\,\mathrm{d}A$$
$$+ \int_{A_\eta} \mathbf{n}_\eta \cdot [-\mathbf{I}(\langle p_\beta\rangle^\beta - \langle p_\beta\rangle_\eta^\beta) + \mu_\beta(\varepsilon_\beta^{-1}\nabla\langle \mathbf{v}_\beta\rangle - \nabla\langle \mathbf{v}_\beta\rangle_\eta)]\,\mathrm{d}A \quad (67)$$

and an *excess bulk stress* which is given by

$$\int_{A_{\omega\eta}} \mathbf{n}_{\omega\eta} \cdot (\langle \mathbf{T}\rangle_B)\,\mathrm{d}A = \int_{V_\omega} \mu_\beta(\mathbf{\Phi}_\beta\mathbf{K}_{\beta\omega}^{-1} \cdot \langle \mathbf{v}_\beta\rangle_\omega)\,\mathrm{d}V + \int_{V_\eta} \mu_\beta(\mathbf{\Phi}_\beta)\,\mathrm{d}V. \quad (68)$$

Finally the *excess Brinkman stress* is defined according to

$$\int_{A_{\omega\eta}} \mathbf{n}_{\omega\eta} \cdot (\langle \mathbf{B}\rangle)\,\mathrm{d}A = \int_{\mathscr{V}_\infty} [\mu_\beta\varepsilon_\beta^{-1}(\nabla \ln \varepsilon_\beta)^2\langle \mathbf{v}_\beta\rangle]\,\mathrm{d}V. \quad (69)$$

Use of these three definitions leads to a momentum jump condition of the form

B.C.1
$$\mathbf{n}_{\omega\eta} \cdot [-\mathbf{I}(\langle p_\beta\rangle_\omega^\beta - \langle p_\beta\rangle_\eta^\beta) + \mu_\beta(\varepsilon_{\beta\omega}^{-1}\nabla\langle \mathbf{v}_\beta\rangle_\omega - \nabla\langle \mathbf{v}_\beta\rangle_\eta)] = \nabla_s \cdot (\delta\langle \mathbf{T}\rangle_S) - \mathbf{n}_{\omega\eta} \cdot (\langle \mathbf{T}\rangle_B) + \mathbf{n}_{\omega\eta} \cdot (\langle \mathbf{B}\rangle) \quad \text{at the } \omega\text{–}\eta \text{ boundary} \quad (70)$$

and various special cases of this result are available [7]. In addition to the boundary condition given by equation (70) we should remind the reader that the second boundary condition is given by

B.C. 2 $$\langle \mathbf{v}_\beta\rangle_\omega = \langle \mathbf{v}_\beta\rangle_\eta \quad \text{at the } \omega\text{–}\eta \text{ boundary.} \quad (71)$$

If the excess surface and bulk stresses are neglected we obtain the following boundary condition

3.5. *Negligible excess stresses*

B.C. 1
$$\mathbf{n}_{\omega\eta} \cdot [-\mathbf{I}(\langle p_\beta\rangle_\omega^\beta - \langle p_\beta\rangle_\eta^\beta) + \mu_\beta(\varepsilon_{\beta\omega}^{-1}\nabla\langle \mathbf{v}_\beta\rangle_\omega - \nabla\langle \mathbf{v}_\beta\rangle_\eta)] = 0 \quad \text{at the } \omega\text{–}\eta \text{ boundary.} \quad (72)$$

This boundary condition would be consistent with the use of equations (46) and (48) and would be referred to as the Brinkman solution [10] for the stress jump condition. For the uniform flow illustrated in Fig. 1, the normal component of equation (72) reduces to

B.C. 1 $$\langle p_\beta\rangle_\omega^\beta = \langle p_\beta\rangle_\eta^\beta \quad \text{at the } \omega\text{–}\eta \text{ boundary.} \quad (73)$$

This result is routinely used without comment and it assumes that both the normal components of the viscous stresses in equation (72) are zero and that the normal components of the excess stresses in equation (70) are negligible.

In the next section we will develop a usable form of the stress jump condition given by equation (70), and there we will see that the complexities associated with the excess stresses can be reduced to a relatively simple form containing a single adjustable parameter. The result will be significantly different from the jump condition of Beavers and Joseph [2] who developed a condition that contained a jump in *both the stress and the velocity*. While the Beavers and Joseph model has enjoyed considerable success in the treatment of fluid mechanics problems, the velocity jump leaves much to be desired in terms of heat transfer processes [25].

4. GENERAL FORM FOR THE EXCESS STRESS

On the basis of the definitions of the excess surface stresses given by equations (67) and (68) we propose the following representations:

4.1. *Excess surface stress*

$$\nabla_s(\delta\langle\mathbf{T}\rangle_s = \mathbf{n}_{\omega\eta}\cdot[-\mathbf{I}(\langle p_\beta\rangle^\beta_\omega-\langle p_\beta\rangle^\beta_\eta) + \mu_\beta(\varepsilon_{\beta\omega}^{-1}\nabla\langle\mathbf{v}_\beta\rangle_\omega-\varepsilon_{\beta\eta}^{-1}\nabla\langle\mathbf{v}_\beta\rangle_\eta)]\cdot\mathbf{C} \quad (74)$$

in which the tensor $\mathbf{C}$ is a dimensionless adjustable tensor coefficient on the order of unity.

4.2. *Excess bulk stress*

$$\mathbf{n}_{\omega\eta}\cdot\langle\mathbf{T}\rangle_B = \delta\mu_\beta\mathbf{D}\cdot[\mathbf{K}_{\beta\omega}^{-1}\cdot\langle\mathbf{v}_\beta\rangle_\omega] \quad (75)$$

in which $\mathbf{D}$ is a dimensionless adjustable tensor coefficient on the order of unity. In order to clarify the choice of these two representations, we consider the one-dimensional example based on the function $\langle\Psi_\beta\rangle$ that has the following property:

$$\langle\Psi_\beta\rangle = \begin{cases}\langle\Psi_\beta\rangle_\omega & y\leqslant-\delta/2\\ \langle\Psi_\beta\rangle_\eta & y\geqslant+\delta/2\end{cases}. \quad (76)$$

The *excess function* associated with $\langle\Psi_\beta\rangle$ is defined by

$$\delta\langle\Psi_\beta\rangle_{ex} = \int_{-\delta/2}^{0}(\langle\Psi_\beta\rangle-\langle\Psi_\beta\rangle_\omega)\,dy + \int_{0}^{\delta/2}(\langle\Psi_\beta\rangle-\langle\Psi_\beta\rangle_\eta)\,dy. \quad (77)$$

The form of this excess function is, of course, identical to the definitions given by equations (67)–(69). In Fig. 5 we have plotted $\langle\Psi_\beta\rangle$, $\langle\Psi_\beta\rangle_\omega$ and $\langle\Psi_\beta\rangle_\eta$ and from that figure we can see that $\delta\langle\Psi_\beta\rangle_{ex}$ is related to the difference between the two shaded areas, i.e. A_1–A_2. We can represent the excess function according to

$$\delta\langle\Psi_\beta\rangle_{ex} = C(\langle\Psi_\beta\rangle_\omega|_{y=0}-\langle\Psi_\beta\rangle_\eta|_{y=0}) \quad (78)$$

in which C is an adjustable parameter. It should be clear that the difference between the two extrapolated values of $\langle\Psi_\beta\rangle$ is an appropriate *scaling factor*; however, it does not necessarily represent the *complete* functional dependence of $\delta\langle\Psi_\beta\rangle_{ex}$.

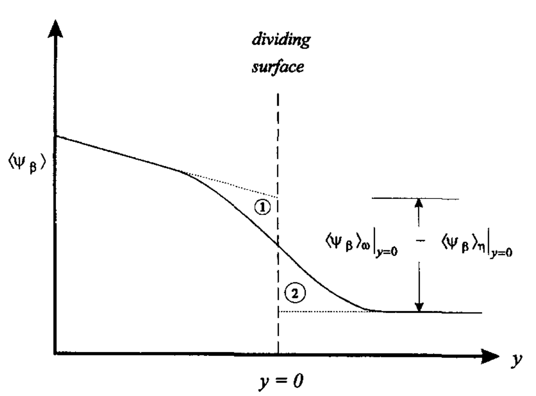

Fig. 5. Determination of an excess function.

4.3. *Excess Brinkman stress*

Extracting a plausible representation for the excess Brinkman stress presents a problem because of the highly non-linear dependence upon the void fraction. If we restrict our thoughts to the tangential component of the velocity we can argue that $\varepsilon_\beta^{-3}\langle\mathbf{v}_\beta\rangle$ is a slowly varying function of position and this encourages us to express the integrand in equation (70) as

$$\mu_\beta\varepsilon_\beta^{-1}(\nabla\ln\varepsilon_\beta)^2\langle\mathbf{v}_\beta\rangle = \mu_\beta(\nabla\varepsilon_\beta)^2(\varepsilon_\beta^{-3}\langle\mathbf{v}_\beta\rangle). \quad (79)$$

The idea that $\varepsilon_\beta^{-3}\langle\mathbf{v}_\beta\rangle$ is a slowly varying function is based on the form of the Blake–Kozeny correlation [26] and it suggests a representation of the excess Brinkman stress given by

$$\mathbf{n}_{\omega\eta}\cdot\langle\mathbf{B}\rangle = \mu_\beta\delta^{-1}\mathbf{A}\cdot(\varepsilon_{\beta\omega}-\varepsilon_{\beta\eta})^2(\varepsilon_{\beta\omega}^{-3}\langle\mathbf{v}_\beta\rangle_\omega+\varepsilon_{\beta\eta}^{-3}\langle\mathbf{v}_\beta\rangle_\eta) \quad (80)$$

in which $\varepsilon_{\beta\eta}^{-3}$ has only been included for clarity since this term is equal to one.

Since the form of the excess surface stress is identical to the left hand side of equation (70), the effect of $\nabla_s\cdot(\delta\langle\mathbf{T}\rangle_s)$ will be lost in the stress jump condition because of the adjustable nature of the coefficients in equations (75) and (80), thus we can substitute these latter two equations into equation (70) to obtain

$$\mathbf{n}_{\omega\eta}\cdot[-\mathbf{I}(\langle p_\beta\rangle^\beta_\omega-\langle p_\beta\rangle^\beta_\eta)+\mu_\beta(\varepsilon_{\beta\omega}^{-1}\nabla\langle\mathbf{v}_\beta\rangle_\omega-\nabla\langle\mathbf{v}_\beta\rangle_\eta)] = -\mu_\beta\delta\mathbf{D}\cdot[\mathbf{K}_{\beta\omega}^{-1}\cdot\langle\mathbf{v}_\beta\rangle_\omega] + \mu_\beta\delta^{-1}\mathbf{A}\cdot(\varepsilon_{\beta\omega}-1)^2(\varepsilon_{\beta\omega}^{-3}\langle\mathbf{v}_\beta\rangle_\omega+\langle\mathbf{v}_\beta\rangle_\eta) \quad \text{at the } \omega\text{–}\eta \text{ interface.} \quad (81)$$

Here we have set $\varepsilon_{\beta\eta}$ equal to one and at this point we are ready to extract the tangential component of the jump condition. This is given by

$$\mathbf{n}_{\omega\eta}\cdot[-\mathbf{I}(\langle p_\beta\rangle^\beta_\omega-\langle p_\beta\rangle^\beta_\eta) + \mu_\beta(\varepsilon_{\beta\omega}^{-1}\nabla\langle\mathbf{v}_\beta\rangle_\omega-\nabla\langle\mathbf{v}_\beta\rangle_\eta)]\cdot\lambda = -\mu_\beta\delta\lambda\cdot\mathbf{D}\cdot[\mathbf{K}_{\beta\omega}^{-1}\cdot\langle\mathbf{v}_\beta\rangle_\omega] + \mu_\beta\delta^{-1}\lambda\cdot\mathbf{A}\cdot(\varepsilon_{\beta\omega}-1)^2(\varepsilon_{\beta\omega}^{-3}\langle\mathbf{v}_\beta\rangle_\omega+\langle\mathbf{v}_\beta\rangle_\eta) \quad (82)$$

and can be simplified by noting that $\mathbf{n}_{\omega\eta}\cdot\lambda$ is zero and that the velocities $\langle\mathbf{v}_\beta\rangle_\omega$ and $\langle\mathbf{v}_\beta\rangle_\eta$ are equal as indicated by equation (71). This leads to the following form:

$$\mathbf{n}_{\omega\eta}\cdot(\varepsilon_{\beta\omega}^{-1}\nabla\langle\mathbf{v}_\beta\rangle_\omega-\nabla\langle\mathbf{v}_\beta\rangle_\eta)\cdot\lambda = [\delta^{-1}\lambda\cdot\mathbf{A}(\varepsilon_{\beta\omega}-1)^2(\varepsilon_{\beta\omega}^{-3}+1) - \delta\lambda\cdot\mathbf{D}\cdot\mathbf{K}_{\beta\omega}^{-1}]\cdot\langle\mathbf{v}_\beta\rangle_\omega. \quad (83)$$

If we scale the thickness of the interfacial region according to $\delta = \mathbf{O}\sqrt{(K_{\beta\omega})}$, we can express the tangential component of our jump condition for the stress as

$$\mathbf{n}_{\omega\eta}\cdot(\varepsilon_{\beta\omega}^{-1}\nabla\langle\mathbf{v}_\beta\rangle_\omega - \nabla\langle\mathbf{v}_\beta\rangle_\eta)\cdot\lambda = \frac{\mathbf{d}}{\sqrt{(K_{\beta\omega})}}\cdot\langle\mathbf{v}_\beta\rangle_\omega \tag{84}$$

in which $\mathbf{d}$ is a dimensionless *drag vector* defined by

$$\mathbf{d} = \sqrt{(K_{\beta\omega})}[\delta^{-1}\lambda\cdot\mathbf{A}(\varepsilon_{\beta\omega}-1)^2(\varepsilon_{\beta\omega}^{-3}+1) - \delta\lambda\cdot\mathbf{D}\cdot\mathbf{K}_{\beta\omega}^{-1}]. \tag{85}$$

For the process illustrated in Fig. 1, we summarize the boundary conditions as

B.C. 1

$$\varepsilon_{\beta\omega}^{-1}\frac{\partial\langle v_\beta\rangle_\omega}{\partial y} - \frac{\partial\langle v_\beta\rangle_\eta}{\partial y} = \frac{\beta}{\sqrt{(K_{\beta\omega})}}\langle v_\beta\rangle_\omega \quad y=0 \tag{86}$$

B.C. 2

$$\langle v_\beta\rangle_\omega = \langle v_\beta\rangle_\eta \quad y = 0. \tag{87}$$

Here we have used $\langle v_\beta\rangle_\omega$ and $\langle v_\beta\rangle_\eta$ to represent the x-components of the two volume average velocity vectors and the dimensionless coefficient β is given by

$$\beta = \sqrt{(K_{\beta\omega})}[\delta^{-1}\lambda\cdot\mathbf{A}\cdot\lambda(\varepsilon_{\beta\omega}-1)^2(\varepsilon_{\beta\omega}^{-3}+1) - \delta\lambda\cdot\mathbf{D}\cdot\mathbf{K}_{\beta\omega}^{-1}\cdot\lambda]. \tag{88}$$

Like the dimensionless drag vector $\mathbf{d}$, we expect the dimensionless coefficient β to be on the order of one, and on the basis of the nature of excess functions as illustrated in Fig. 5, we can expect that β may be either positive or negative. In Part II we will compare solutions of equations (45)–(48), subject to the boundary conditions given by equations (86) and (87) with the experimental data of Beavers and Joseph [2].

5. CONCLUSIONS

In this study we have derived a stress jump condition for the boundary between a porous medium and a homogeneous fluid. The development is based on a generally valid, non-local form of the volume averaged Stokes' equations, and it requires experimental measurements to evaluate an undetermined parameter that naturally appears in the jump condition. The jump condition is constructed in order to connect Darcy's law *with the Brinkman correction* to the Stokes' equations, and this leads to a volume averaged velocity field that is continuous.

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REFERENCES

1. D. D. Joseph and L. N. Tao, Lubrication of a porous bearing—Stokes' solution, *J. Appl. Mech.*, 753–760 (1966).
2. G. S. Beavers and D. D. Joseph, Boundary conditions at a naturally permeable wall, *J. Fluid Mech.* **30,** 197–207 (1967).
3. R. E. Larson and J. J. L. Higdon, Microscopic flow near the surface of two-dimensional porous media—I. Axial flow, *J. Fluid Mech.* **166,** 449–472 (1987).
4. R. E. Larson and J. J. L. Higdon, Microscopic flow near the surface of two-dimensional porous media—II. Transverse flow, *J. Fluid Mech.* **178,** 119–136 (1987).
5. M. Sahraoui and M. Kaviany, Slip and no-slip velocity boundary conditions at interface of porous, plain media, *Int. J. Heat Mass Transfer* **35,** 927–943 (1992).
6. M. Kaviany, *Principles of Heat Transfer in Porous Media.* Springer, New York (1991).
7. D. A. Nield and A. Bejan, *Convection in Porous Media.* Springer, New York (1992).
8. S. Whitaker, The species mass jump condition at a singular surface, *Chem. Engng Sci.* **47,** 1677–1685 (1992).
9. S. Haber and R. Mauri, Boundary conditions for Darcy's flow through porous media, *Int. J. Multiphase Flow* **9,** 561–574 (1983).
10. C. W. Somerton and I. Catton, On the thermal instability of superposed porous and fluid layers, *Trans. ASME* **104,** 160–165 (1982).
11. J. E. Maneval, M. J. McCarthy and S. Whitaker, Use of NMR as an experimental probe in multiphase systems: determination of the instrument weighting function for measurements of liquid-phase volume fraction, *Water Resour. Res.* **26,** 2807–2816 (1990).
12. S. A. Altobelli, R. C. Givler and E. Fukushima, Velocity and concentration measurements of suspensions by nuclear magnetic resonance imaging, *J. Rheol.* **35,** 721–732 (1991).
13. F. A. Howes and S. Whitaker, The spatial averaging theorem revisited, *Chem. Engng Sci.* **40,** 1387–1392 (1985).
14. S. Whitaker, Flow in porous media—I. A theoretical derivation of Darcy's law, *Transp. Porous Media* **1,** 3–25 (1986).
15. H. C. Brinkman, A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles, *Appl. Sci. Res.* **A1,** 27–34 (1947).
16. M. Quintard and S. Whitaker, Transport in ordered and disordered porous media—I. The cellular average and the use of weighting functions, *Transp. Porous Media* **14,** 163–177 (1994).
17. M. Quintard and S. Whitaker, Transport in ordered and disordered porous media—II. Generalized volume averaging, *Transp. Porous Media* **14,** 179–206 (1994).
18. R. G. Carbonell and S. Whitaker, Heat and mass transfer in porous media. In *Fundamentals of Transport Phenomena in Porous Media* (Edited by J. Bear and M. Y. Corapcioglu), pp. 123–198. Martinus Nijhoff, Dordrecht, The Netherlands (1984).
19. S. Whitaker, Advances in the theory of fluid motion in porous media, *I&EC Fundam.* **61,** 14–28 (1969).
20. M. Quintard and S. Whitaker, Transport in ordered and disordered porous media—III. Closure and comparison between theory and experiment, *Transp. Porous Media* **15,** 31–49.
21. M. Quintard and S. Whitaker, Transport in ordered and disordered porous media—IV. Computer generated porous media, *Transp. Porous Media* **15,** 51–70 (1994).
22. M. Quintard and S. Whitaker, Transport in ordered and disordered porous media—V. Geometrical results for two-dimensional systems, *Transp. Porous Media* **15,** 183–196.
23. J. A. Ochoa-Tapia, P. Del Rio and S. Whitaker, Bulk and surface diffusion in porous media: an application of the surface averaging theorem, *Chem. Engng Sci.* **48,** 2061–2082 (1993).
24. J. Douglas and T. Abogast, Dual porosity models for flow in naturally fractured reservoirs. In *Dynamics of*

Fluids in Hierarchical Porous Media (Edited by J. H. Cushman), Chap. VII. Academic Press, New York (1990).
25. K. Vafai and R. Thiyagaaraja, Analysis of flow and heat transfer at the interface region of a porous medium, *Int. J. Heat Mass Transfer* **30,** 1391–1405 (1987).
26. R. B. Bird, W. E. Stewart and E. N. Lightfoot, *Transport Phenomena*. John Wiley, New York (1960).